Bragg solitons and the nonlinear Schrödinger equation

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We develop explicitly the description of low-intensity Bragg solitons by the nonlinear Schrödinger equation. In contrast to earlier studies, this result is established for arbitrary soliton velocities. This result applies directly to grating solitons that have been observed in the laboratory. [S1063-651X(98)11212-6]

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I. INTRODUCTION AND BACKGROUND

Solitons in uniform optical fibers are understood as arising from a balancing of the quadratic fiber dispersion and the Kerr nonlinearity of the glass [1,2]. Such solitons are usually described theoretically by the nonlinear Schrödinger equation (NLSE), in which terms associated with dispersion and nonlinearity explicitly appear. The NLSE has been studied extensively and many of its properties, such as integrability and the robustness of its solutions [3], can be applied to fiber solitons.

It was shown more recently [4–11] that Bragg solitons can exist in fiber Bragg gratings. In these solitons, which can propagate at any velocity between 0 and V, the speed of light in unprocessed fiber [5–7], the glass nonlinearity balances the grating dispersion. Nonetheless, pulse propagation in fiber gratings is governed by a set of nonlinear coupled mode equations (CMEs), not by the NLSE [5–7]. The relation between the NLSE and the more general CME description, which was discussed earlier [12,13], is important. If it is established that the NLSE applies in some limit, then its properties apply to Bragg solitons. This was established explicitly only for low-velocity and low-intensity solitons [12]. However, the Bragg solitons that are most easily generated in the laboratory travel at 60–80% of V and for these solitons, observed by Eggleton *et al.* [9,10], the existing work is insufficient. Here we rectify this and show explicitly that the NLSE applies in the low-intensity limit, but for *any* velocity.

Bragg solitons are described by the CMEs [5,7]

$$\frac{i}{V}\frac{\partial E_\pm}{\partial t} \pm i\frac{\partial E_\pm}{\partial z} + \kappa E_\mp + (\Gamma_S|E_\pm|^2 + 2\Gamma_\times|E_\mp|^2)E_\pm = 0, \quad (1)$$

where $E_\pm$ are the amplitudes of the forward and backward propagating modes, V is the group velocity in unprocessed fiber, κ is a coupling coefficient describing the grating strength, and $\Gamma_{S,\times}$ are, respectively, self- and cross-phase modulation parameters. Expressions for these parameters are known [5], but are not given here.

Among the known solutions to Eqs. (1) are those of Aceves and Wabnitz [7], who found a two-parameter family of pulselike solutions. Their solutions are considered the most general description of Bragg solitons and have two free parameters: $\mathcal{V}$ is the soliton's velocity in units of V and δ represents its amplitude and center frequency. Below we shall refer to solutions with $\delta \ll 1$ as *low-intensity* Bragg solitons. We note that for grating solitons that have been observed in the laboratory by Eggleton *et al.*, $\delta \approx 0.10$ [9], so that these have a low intensity by this definition, even though it exceeds 10 GW/cm^2.

The dispersion relation of the linearized CMEs was discussed before [5]. Briefly, the photonic bands can be described using the relative group velocity v as a fraction of V (hence $-1<v<1$). In terms of v,

$$\Omega_\pm = \pm V\kappa\gamma, \quad Q = \kappa\gamma v, \quad (2)$$

where $\gamma = 1/\sqrt{1-v^2}$ and Ω and Q are the frequency and wave number with respect to the Bragg resonance, respectively. According to Eqs. (2), $\Omega^2/V^2 + Q^2 = \kappa^2$, which is the way the dispersion relation is usually written. The associated linear eigenstates, indicated by $|\pm\rangle$, are also well known [5].

II. MULTIPLE-SCALE ANALYSIS

The NLSE is derived from the CMEs (1) using the method of multiple scales [12,13]. The key to this analysis is the introduction of coordinates describing phenomena occurring at different length and time scales through [12] $z = z_0 + \mu z_1 + \mu^2 z_2 + \cdots$, where z is position, and similarly for the time t. Since $\mu \ll 1$, z_0, and t_0 describe phenomena on the shortest length and time scales. Longer length and time scales are described by z_1, t_1 and z_2, t_2. Henceforth all these parameters are considered independent of each other.

To describe solutions to Eqs. (1) with a center frequency larger than the Bragg frequency of the grating (this includes low-intensity grating solitons in a medium with positive nonlinearity), we write the $E_\pm$ as [12]

$$\mathbf{E} = \mu(a|+\rangle + \mu b|-\rangle)e^{i\kappa\gamma(vz_0 - Vt_0)}, \quad (3)$$

where $\mathbf{E}$ indicates the vector (E_+, E_-). Further, $a = a(z_1, z_2; t_1, t_2)$ and $b = b(z_1, z_2; t_1, t_2)$ vary slowly on the scale of Ω and Q since they do not depend on z_0 or t_0.

The ansatz (3) is now substituted into Eqs. (1) and contributions at increasing higher order of μ are collected. To order μ^0 it is found that (3) satisfies Eq. (2). To order μ [12]

$$\frac{\partial a}{\partial \tau_1} = 0, \quad b = -\frac{i}{2\kappa\gamma^2}\frac{\partial a}{\partial \zeta_1}, \quad (4)$$

where we introduced

$$\zeta_1 = z_1 - vVt_1, \quad \tau_1 = t_1. \tag{5}$$

Thus, to this order, a travels at the group velocity following from Eq. (2).

Taking the procedure to order μ^2, it is found that

$$\frac{i}{V}\frac{\partial a}{\partial \tau_2} + \frac{i}{\gamma}\frac{\partial b}{\partial \zeta_1} + \frac{1}{2}(\Gamma_+ + \Gamma_- v^2)|a|^2 a = 0, \tag{6}$$

$$i\frac{\partial a}{\partial \zeta_2} - 2i\gamma v\frac{\partial b}{\partial \zeta_1} + \frac{i\gamma}{V}\frac{\partial b}{\partial \tau_1} + \frac{v}{2}\Gamma_-|a|^2 a = 0, \tag{7}$$

where ζ_2 and τ_2 were defined similarly to Eqs. (5) and

$$\Gamma_\pm = \Gamma_S \pm 2\Gamma_\times. \tag{8}$$

From Eqs. (6) and (4) we find that a obeys the NLSE

$$i\frac{\partial a}{\partial \tau_2} + \frac{\Omega''}{2}\frac{\partial^2 a}{\partial \zeta_1^2} + \frac{1}{2}(\Gamma_+ + \Gamma_- v^2)|a|^2 a = 0, \tag{9}$$

where the quadratic dispersion $\Omega'' = V/\kappa\gamma^3$ can be found from Eqs. (2) with the positive sign. The result (9) was obtained earlier in Refs. [12] and [13].

We now combine Eqs. (7) and (4) to find that

$$i\frac{\partial a}{\partial \zeta_2} - \frac{v}{\kappa\gamma}\frac{\partial^2 a}{\partial \zeta_1^2} + \frac{v}{2}\Gamma_-|a|^2 a = 0. \tag{10}$$

This equation was not noted before and is the key to this paper. It gives the dependence of envelope function a on the longest length scale (ζ_2) [note that the NLSE (9) does not depend on ζ_2]. When $v = 0$, a does not depend on ζ_2 and Eq. (10) is thus irrelevant, so that its absence was not noted earlier [12,13].

Now *any* solution to the NLSE (9) leads, via Eq. (10), to an approximate solution to CMEs (1) with spectral content at a frequency above the Bragg frequency of the grating. However, to illustrate the application of our results, we consider the specific example of the one-soliton solution of the NLSE. In our notation it reads

$$\tilde{a} = C\sqrt{\frac{2\kappa\gamma^3}{\Gamma_+ + \Gamma_- v^2}}\, e^{iC^2\tau_2\kappa\gamma^3/2}\,\mathrm{sech}(C\kappa\gamma^3\zeta_1), \tag{11}$$

where C is a free parameter. We have used $\tilde{a}$ to indicate that the ζ_2 dependence through Eq. (10) has not yet been included. Equation (11) contains two free parameters C and v. We now define

$$\nu = C\gamma^2. \tag{12}$$

Note that when $v \to 0$, the parameters ν and C are identical. Equation (11) can now be written as

$$\tilde{a} = \alpha\nu\sqrt{\frac{\kappa\gamma}{\Gamma_\times}}\, e^{i\nu^2\tau_2\kappa/(2\gamma)}\,\mathrm{sech}(\nu\kappa\gamma\zeta_1), \tag{13}$$

where

$$\alpha^{-2} = 1 + \frac{\Gamma_s}{2\Gamma_\times}\frac{1+v^2}{1-v^2}. \tag{14}$$

Though Eq. (13) is an exact solution to the NLSE (9), for the construction of approximate solutions to Eqs. (1) we must consider ν to be small. Below, therefore, we drop terms of order ν^3 and higher.

To obtain a, we require Eq. (10). Using Eq. (13) it is straightforward to show that

$$a = \tilde{a}\exp\left[(i\kappa\gamma\nu\zeta_2)\nu\left(\frac{2\Gamma_s v}{\Gamma_+ + v^2\Gamma_-}\right)\right], \tag{15}$$

where terms of order ν^3 and higher were dropped. Note that this phase factor originates from relation (10) and does not follow directly from earlier work [12,13]. It corresponds to a similar factor in the Aceves-Wabnitz solution [7]. Using Eq. (4) one can also show that

$$b = \frac{i\nu}{2\gamma}\tanh(\kappa\nu\gamma\zeta_1)a, \tag{16}$$

so that, according to Eq. (3),

$$\mathbf{E} = \frac{a}{\sqrt{2}}\begin{pmatrix} (1+v)^{1/2}[1 + i\nu\tanh(\nu\kappa\gamma\zeta_1)/2] \\ -(1-v)^{1/2}[1 - i\nu\tanh(\nu\kappa\gamma\zeta_1)/2] \end{pmatrix} e^{iv\nu\tanh(\nu\kappa\gamma\zeta_1)/2}. \tag{17}$$

The solution (17) is identical to the exact solutions of Aceves and Wabnitz [7] if we identify δ and $\mathcal{V}$, respectively, with ν and v and when terms of order δ^3 and higher are ignored. This shows that, using the Eq. (10), we can construct solutions to the CMEs (1) from NLSE (9). Though here we illustrated this with the NLSE's one-soliton solution, we stress that any solution to the NLSE (9), for example, any cw soluton, or any periodic solution leads to an approximate solution to CMEs (1).

III. DISCUSSION AND CONCLUSIONS

Thus we have tightened a loose end in the literature: Generalizing earlier work, we show explicitly how a solution of the CMEs may be found from a solution of the NLSE. We use this to show that in the low-intensity limit though for any soliton velocity, the solutions of Aceves and Wabnitz [7] can be constructed from one-soliton solutions of the NLSE. We note that for the grating solitons observed experimentally by Eggleton *et al.*, $\delta \approx 0.10$, so that, according to our definition, they have low intensity. They propagate at roughly 75% of the speed of light in unprocessed fiber [9]. They are thus precisely the class of solitons to which the results derived here apply. We note further that Eq. (12) shows that the standard parametrization of the NLSE is not appropriate for grating solitons. One must make use of Eq. (12) to relate the solutions to these two equations.

The key result is Eq. (10), which leads to the phase factor (15). It formally justifies the simple intuitive description of low-intensity Bragg solitons in terms of the balancing of the quadratic grating dispersion and the fiber nonlinearity. It also establishes that, within the limits discussed, NLSE properties such as soliton robustness may be applied to Bragg solitons.

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